

A **related rates** problem involves finding the unknown rate of change of one quantity by relating it to the already known rates of change of one or more other quantities. The following example involves relating rates of change that occur with respect to time. It is possible to relate rates of change that occur with respect to a quantity other than time.

Example: “A man starts walking north at 4 ft/s from a point P. Five minutes later a woman starts walking south at 5 ft/s from a point 500 ft due east of P. At what rate are the [two] moving apart 15 minutes after the woman starts walking?” (Stewart 203)

1. **Assign symbols to all quantities that are functions of time.** If possible **draw a diagram** that describes the situation, and that includes symbols and given constant values (i.e. 500 ft).

x_m = distance walked north by the man

x_w = distance walked south by the woman

h = distance from the man to the woman

Figure 1 (at $t = 5 \text{ min} = 300 \text{ sec}$)

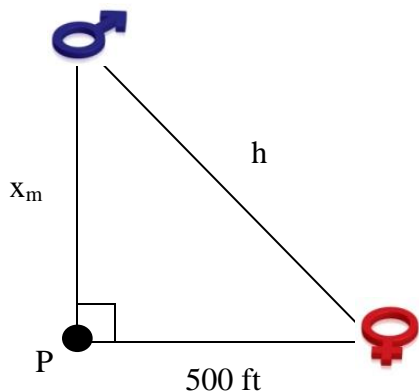
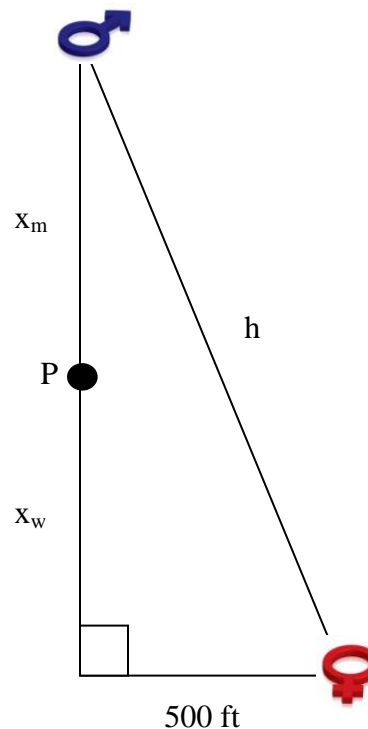


Figure 2 (at $t = 20 \text{ min} = 1200 \text{ sec}$)



2. **Express the given rate(s) AND the unknown rate in terms of derivatives**, with respect to time, t .

$$\frac{dx_m}{dt} = 4 \frac{\text{ft}}{\text{s}}$$

$$\frac{dx_w}{dt} = 5 \frac{\text{ft}}{\text{s}}$$

$$\frac{dh}{dt} = ? \frac{\text{ft}}{\text{s}}$$

3. **Find an equation that relates the quantities** of the given rate(s) and the unknown rate. If possible, use the geometry of the situation and relationships between quantities to **eliminate variables by substitution**.

$$(x_m + x_w)^2 + 500^2 = h^2 \qquad x_m^2 + 2x_mx_w + x_w^2 - h^2 = -500^2$$

4. **Differentiate both sides** of the equation with respect to time, t .

$$2x_m \frac{dx_m}{dt} + 2x_w \frac{dx_w}{dt} + 2x_w \frac{dx_m}{dt} + 2x_m \frac{dx_w}{dt} - 2h \frac{dh}{dt} = 0$$

5. Place the given information into the equation and **solve for the unknown rate**.

$$\frac{dh}{dt} = \left(2x_m \frac{dx_m}{dt} + 2x_w \frac{dx_w}{dt} + 2x_w \frac{dx_m}{dt} + 2x_m \frac{dx_w}{dt} \right) \frac{1}{2h}$$

$$x_m = \left(4 \frac{\text{ft}}{\text{s}} \right) (1200 \text{ s}) = 4800 \text{ ft} \qquad x_w = \left(5 \frac{\text{ft}}{\text{s}} \right) (900 \text{ s}) = 4500 \text{ ft}$$

$$h = \sqrt{500^2 + (4800 + 4500)^2} \approx 9313 \text{ ft}$$

$$\frac{dh}{dt} \approx 8.99 \frac{\text{ft}}{\text{s}}$$

Practice Problems

- “A particle is moving along the curve $y = x^{1/2}$. As the particle passes through the point $(4, 2)$, its x -coordinate increases at a rate of 3 cm/s. How fast is the distance from the particle to the origin changing at this instant?” (Stewart 203)
- “Gravel is being dumped from a conveyor belt at a rate of 30 ft³/min, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?” (Stewart 203)

Solutions:

1. 3.02 cm/s

2. $6/(5\pi) \approx 0.38$ ft/min