Partial Fractions *Demystified*

Here is a **Five step process** for using Partial Fractions to integrate indefinite integrals of the form

$$\int \frac{P(x)}{Q(x)}\,dx, \quad \text{where } P(x) \text{ and } Q(x) \text{ are polynomials.}$$

STEP(1) Compare the degrees of $P(x)$ and $Q(x)$.

(a) If $deg(P) \geq deg(Q)$, then use polynomial long division and proceed to step(2).
(b) If $deg(P) < Deg(Q)$, we're in good shape and proceed to step(2).

STEP(2) Factor the denominator $Q(x)$.

As discussed in Stuart, $Q(x)$ will have four possible factor types: (i) distinct linear factors (ii) repeated linear factors (iii) distinct irreducible quadratic factors or (iv) repeated irreducible quadratic factors.

STEP(3) Determine which of the above cases are present in your factorization and build the pieces.

Ask yourself, (1) Are there Linear Factors? If yes, then are they (a) Distinct or (b) Repeated? Next, (2) Are there Irreducible Quadratic Factors? If yes, then are they (a) Distinct or (b) Repeated? For each case, set up the appropriate partial fraction and save for later.

STEP(4) Put the pieces you saved for later all together and calculate the constants.

STEP(5) Integrate.

Now that $\frac{P(x)}{Q(x)}$ has been broken up into partial fractions, go ahead and integrate each case seperately. When the integration is complete, put it all together for a final answer.

Warning: DO NOT FORGET THE CONSTANT OF INTEGRATION!

EXAMPLE

$$\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\,dx$$

STEP(1) Compare $deg(P)$ with $deg(Q)$.	Here $deg(P)=2<4=deg(Q)$, where $P(x)=x^2-2x-1$ and $Q(x)=(x-1)^2(x^2+1)$. So polynomial long division is not necessary.
STEP(2) Factor the denominator.	It's already factored for us, so we move on.
STEP(3) Case check.	(1) Are there Linear factors? Yes, $(x-1)^2$ Is it (a) Distinct or (b) Repeated? It is Repeated. Use case (ii) to break it up. SET UP: Use $\frac{A}{(x-1)}+\frac{B}{(x-1)^2}$, save for later. (2) Are there Quadratic factors? Yes, x^2+1. Is it (a) Distinct or (b) Repeated? It is Distinct. Use case (iii) to break it up. SET UP: Use $\frac{Cx+D}{x^2+1}$, save for later.
STEP(4) Put it all together and calculate the constants.	We now have, $\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+1}=\frac{x^2-2x-1}{(x-1)^2(x^2+1)}$. We go ahead and compute the constants and find that $A=1,B=-1,C=-1,D=1$. Therefore, $\frac{x^2-2x-1}{(x-1)^2(x^2+1)}=\frac{1}{(x-1)}+\frac{-1}{(x-1)^2}+\frac{-x+1}{x^2+1}$ and thus our integral can be written as $\int\frac{x^2-2x-1}{(x-1)^2(x^2+1)}dx=\int\frac{1}{(x-1)}dx+\int\frac{-1}{(x-1)^2}dx+\int\frac{-x+1}{x^2+1}dx$
STEP(5) Integrate.	As you can see the left side has been broken up into three smaller pieces on the right hand side. So now all thats left to do is integrate each piece separately and put the results together for the final answer.

PRACTICE PROBLEMS

	Solutions
(1) $\int \frac{2x^3-x^2+2x-2}{x^4-x^3}\,dx$	(1) $ln\,\lvert x^2-x\rvert - x^2 + C$
(2) $\int \frac{-12}{(x^2+3)(x^2-9)}\,dx$	(2) $\frac{1}{\sqrt{3}}\,tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+\frac{1}{6}\,ln\left\lvert\frac{x+3}{x-3}\right\rvert + C$
(3) $\int \frac{x+5}{6x^2-13x-5}\,dx$	(3) $-\frac{14}{51}\,ln\,\lvert 3x+1\rvert+\frac{15}{34}\,ln\,\lvert 2x-5\rvert + C$